

# Quantum Stern-Gerlach experiment and path entanglement of Bose-Einstein condensate

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**Abstract** – In this paper, a quantum Stern-Gerlach thought experiment is introduced where, in addition to the intrinsic angular momentum of an atom, the magnetic field is also treated quantum mechanically. A freely falling spin polarised Bose-Einstein condensate passes close to a flux-qubit and interacts with the quantum superimposed magnetic field of the flux-qubit. Such an interaction results a macroscopic quantum entanglement of the path of a Bose-Einstein condensate with the magnetic flux quantum state of the flux-qubit. In this paper, three regimes of coupling between the flux-qubit and a freely falling Bose-Einstein condensate are discussed. The decoherence time limit required to achieve a strong coupling regime is also estimated. This paper also explores, how to produce a path entangled Bose-Einstein condensate where, the condensate can be located at physically distinct locations simultaneously. Paper provides new fundamental insights about the foundations of the quantum Stern-Gerlach experiment.

**Introduction.** – The Stern-Gerlach experiment has a fundamental significance as it provides a clear evidence of quantization of intrinsic momentum of an atom (historically known as space-quantization) [1–3]. The classic Stern-Gerlach experiment is regarded as one of the most important and extensively explored experiments of physics [4–13]. In the context of foundations of physics, the Stern-Gerlach experiment is widely regarded as a precursor of thought experiments. According to quantization of intrinsic angular momentum, the component of angular momentum of a particle about an arbitrary fixed axis (also known as to be the quantization axis) acquires discrete values. The Stern-Gerlach apparatus consists of magnets producing a stronger magnetic field gradient along the quantization axis and the collimated atoms are passed through the magnetic field gradient region, where the stronger magnetic field gradient direction is perpendicular to initial velocity direction of atoms. A neutral atom having a nonzero magnetic-moment component along the quantization axis experiences a force perpendicular to its initial velocity direction. Such a force deviates the path of the atom passing through the Stern-Gerlach apparatus. As the same component of the magnetic-moment and the angular mo-

mentum of the atom can be determined simultaneously therefore, the deviation of the path of the atom is proportional to the projection of angular momentum and the magnetic field gradient along the quantization axis. Since the projection of angular momentum along the quantization axis acquires discrete values therefore, the resultant force experienced by the atom, during its passage through the region of magnetic field gradient, is quantised. The quantized force produces discreteness in the paths of the atom. On the other hand, if an atom is initially prepared in a quantum superposition state of angular momentum components along the quantization axis then, after passing through the Stern-Gerlach apparatus, all of the possible atomic paths are quantum entangled with the angular momentum components. In the case of many such mutually non-interacting atoms passing through the Stern-Gerlach apparatus, the resulting quantum state remains a product state of individual quantum states of all atoms where the quantum entanglement persists at a single atom level. Another important consideration of the Stern-Gerlach experiment is that it is the spin degree of freedom of the atom which is quantised, while the magnetic field is considered to be a classical field with a well defined classical magnetic

field gradient. Since the magnetic field is considered to be the classical field therefore, such a Stern-Gerlach experiment is a semi-classical experiment.

In this paper, a quantum version of the semi-classical Stern-Gerlach experiment has been presented where, in addition to the spin of an atom, the magnetic field and the magnetic field gradient are governed by quantum mechanics. In the quantum Stern-Gerlach experiment, the magnetic field can be prepared in a quantum superposition state. The quantum nature of the magnetic field has remarkable consequences, for example, if a spin polarized Bose-Einstein condensate [14–17] is passed through a quantum Stern-Gerlach apparatus then the path of the Bose-Einstein condensate can be quantum entangled with the magnetic field. Such an entangled quantum state is a multi-particle macroscopic entangled quantum state or a Schrödinger-cat state [18], while in the case of a semi-classical Stern-Gerlach experiment the quantum entanglement exists at a single particle level only. In this paper, starting from a proof-of-principle idea of the quantum Stern-Gerlach experiment, the experimental feasibility of the quantum Stern-Gerlach experiment is explored. This paper also highlights fundamental conditions which should be full-filled to realise the quantum Stern-Gerlach experiment and a path entangled Bose-Einstein condensate.

**Stern-Gerlach experiment and quantum entanglement.** – Consider a neutral atom of total spin  $\mathbf{F}$  (which includes the nuclear spin) is passed through a semi-classical Stern-Gerlach apparatus, which has a predominant component of the magnetic field and a predominant component of its gradient along the  $z$ -axis. The  $z$ -axis is also considered to be the quantization axis. Before entering the Stern-Gerlach apparatus, the atom is moving along the  $x$ -axis with a zero expectation value of its transverse momentum. The projection of the angular momentum along the quantization axis is  $m_F \hbar$ , where  $m_F$  is an integer that varies from  $-F$  to  $+F$  and  $\hbar (h/2\pi)$  is the reduced Plank 's constant. Therefore, the total quantum state of the moving atom at time  $t$  in the centre of mass frame is  $|\psi(\mathbf{r}; t)\rangle \sum_{m_F=-F}^F c_{m_F}(t) |m_F\rangle$ , where  $|\psi(\mathbf{r}; t)\rangle$  is the initial quantum state, in the external degrees of freedom, of the atom in the centre of mass frame prior to its interaction with the magnetic field. In this paper, the bold symbols placed in the argument of ket vectors denote the bases in which a ket vector is expressed except the time parameter  $t$ , which is separated by a semicolon. Therefore,  $|\psi(\mathbf{r}; t)\rangle = \int \psi(r, t) |r\rangle dr$ , where  $|r\rangle$  is the position basis and  $\psi(r, t) = \langle r | \psi(\mathbf{r}; t) \rangle$  is the spatial wavefunction. The quantum state  $\sum_{m_F=-F}^F c_{m_F}(t) |m_F\rangle$  represents the state of the spin degrees of freedom of an atom,  $|m_F\rangle$  is the quantum state corresponding to the projection of spin along the quantization axis with probability amplitude  $c_{m_F}(t)$ . As soon as the atom enters the magnetic field gradient region, a quantized force acts on the atom, which imparts to the atom a spin projection dependent momentum along the  $z$ -axis. The interaction term of the

Hamiltonian of an atom of magnetic moment  $\boldsymbol{\mu}$  in the presence of magnetic field  $\mathbf{B}(r)$  is  $-\boldsymbol{\mu} \cdot \mathbf{B}(r)$ . Therefore, the force acting on the atom is  $\mathbf{f} = \nabla(\boldsymbol{\mu} \cdot \mathbf{B}(r))$ , which can also be expressed as  $\mathbf{f} = -g_F \mu_B \nabla(\mathbf{F} \cdot \mathbf{B}(r)) / \hbar$ , where  $g_F$  is the Landé g-factor and  $\mu_B$  is the Bohr magneton. The magnetic moment and the total spin are related through  $\boldsymbol{\mu} = -g_F \mu_B \mathbf{F} / \hbar$ . The atom interacts with the magnetic field for a time  $\Delta t$  during its passage through the Stern-Gerlach apparatus. Therefore, the transverse linear momentum imparted to the atom, along the  $z$ -axis, is  $p_z = -m_F g_F \mu_B \frac{\partial B_z}{\partial z} \Delta t$ . The transverse linear momentum imparted to the atom is quantized because it is directly proportional to the quantised projection of the spin angular momentum along the quantization axis. Higher is the magnetic field gradient more is the transverse linear momentum splitting. Consider an atom in its initial state  $|\psi(\mathbf{r}; t)\rangle \sum_{m_F=-F}^F c_{m_F}(t) |m_F\rangle$ . After the interaction time  $\Delta t$ , if the difference of the imparted transverse linear momenta corresponding to successive  $m_F$  is greater than the uncertainty of the linear momentum component, along the  $z$ -direction, of the atom 's initial wave-function  $\psi(r, t)$ , then the total quantum state of the atom becomes

$$|\alpha(\mathbf{r}, \mathbf{m}_F; t)\rangle_a = \sum_{m_F=-F}^F \left( \int \Psi(r, m_F, t) |r\rangle dr \right) \times c_{m_F}(t) |m_F\rangle \quad (1)$$

The probability amplitude  $\Psi(r, m_F, t)$  is the spatial wave-function of the atom with an imparted transverse linear momentum  $p_z$  corresponding to  $m_F$ . Denote the term given in the bracket of Eq. 1 *i.e.*  $\int \Psi(r, m_F, t) |r\rangle dr$  with  $|\psi_{p_z}(\mathbf{r}; t)\rangle$ , which corresponds to a quantum state of an atom with an imparted transverse linear momentum  $p_z$ . Consider the magnetic field gradient strength is sufficient to splits the atomic paths corresponding to each transverse linear momentum imparted to the atom *i.e.*  $\langle \psi_{p'_z}(\mathbf{r}; t) | \psi_{p_z}(\mathbf{r}; t) \rangle = 0$ , where  $p_z$  corresponds to a given  $m_F$  and  $p'_z$  corresponds to  $m'_F$  and  $m'_F$  is an allowed integer nearest to  $m_F$ . Therefore, the quantum state given in Eq. 1 is a single atom entangled quantum state *i.e.* the wave-function  $\Psi(r, m_F, t)$  cannot be written as a product of a wave-function of  $m_F$  and a wave-function of space variables. This signifies that the spin angular momentum projection along the quantization axis is quantum entangled with the transverse linear momentum or the path of the atom. Because a quantised force acts on each atom independent of the quantum state of the other atoms therefore, if more than one non-interacting atoms are passed through a semi-classical Stern-Gerlach apparatus, the total quantum state will be a product state of a single atom entangled quantum state given in Eq. 1. In the semi-classical Stern-Gerlach experiment, as described above, the magnetic field and its gradient ( $\partial B_z / \partial z$ ) are governed by the classical physics having well defined values and it is the intrinsic angular momentum of the atom which is quantised.

### Quantum Stern-Gerlach experiment: Principle.

– The quantum Stern-Gerlach experiment, in addition to the quantization of intrinsic angular momentum of the atom, treats the magnetic field quantum mechanically by incorporating the quantum superposition principle to the magnetic field. The quantum superposition of the magnetic field is produced by a flux-qubit [19–22], which also creates a quantum superimposed magnetic field gradient. The schematic of a quantum Stern-Gerlach experiment is illustrated in Fig. 1, where the source of neutral atoms is a trapped Bose-Einstein condensate. If a single atom is un-trapped from the trap then it falls freely under gravity along the positive  $x$ -direction. Consider the quantum state of spin of the atom to be a quantum superposition of the spin projections *i.e.*  $\sum_{m_F=-F}^F c_{m_F}(t)|m_F\rangle$ . The free falling atom comes in the close proximity of the flux-qubit where it interacts, for a time duration  $\Delta t$ , with the magnetic field produced by the flux-qubit. The atom continues a free fall and as it moves away from the flux-qubit, the interaction between the magnetic field and the atom diminishes. The centre of the closed loop of the flux-qubit is considered to be the origin, where the quantization axis is perpendicular to the plane of the flux-qubit loop along the  $z$ -axis as shown in Fig. 1. The flux-qubit is a superconducting loop interrupted by a single Josephson-junction, where the net magnetic flux passing through the flux-qubit loop is considered to be the macroscopic quantum observable. The Hamiltonian of the flux-qubit is written as,  $H_Q = \frac{p_\Phi^2}{2C_j} + \frac{(\Phi - \Phi_a)^2}{2L} + E_j(1 - \cos(2\pi\Phi/\Phi_o))$  [22–26]. Where,  $\Phi_o = h/2e$  is the magnetic flux quantum,  $e$  is electron charge,  $\Phi$  is the net magnetic flux passing through the flux-qubit loop,  $p_\Phi = -i\hbar\partial/\partial\Phi$  is the momentum operator conjugate to  $\Phi$ ,  $(\Phi - \Phi_a)^2/2L$  is the magnetic energy stored in the flux-qubit loop of self-inductance  $L$ ,  $E_j(1 - \cos(2\pi\Phi/\Phi_o))$  is the potential energy of the Josephson-junction of junction capacitance  $C_j$ , Josephson energy  $E_j = I_c\hbar/2e$  and  $I_c$  is the maximum current that can pass through the Josephson-junction without dissipation. The total potential energy of the flux-qubit has two global minima, if the externally applied magnetic flux  $\Phi_a$  is equal to half of the flux quantum ( $\Phi_a = \Phi_o/2$ ). Therefore, if  $\Phi_a = \Phi_o/2$ , the potential energy profile close to the minima can be considered as a symmetric double-well potential. The potential energy profile becomes asymmetric, if the externally applied magnetic flux deviates from  $\Phi_o/2$ . The tunneling amplitude between the wells is governed by the barrier height  $E_j$ , which can be controlled through an additional external magnetic field by replacing the single Josephson-junction with a dc-SQUID (dc-Superconducting Quantum Interference Device). Therefore, by allowing the tunneling between the potential wells, the flux-qubit can be prepared in the ground state of the symmetric double-well potential. The ground state of such a symmetric double well potential corresponds to a quantum superposition of the persistent current flowing in the clockwise and in the anti-clockwise direction. Therefore,

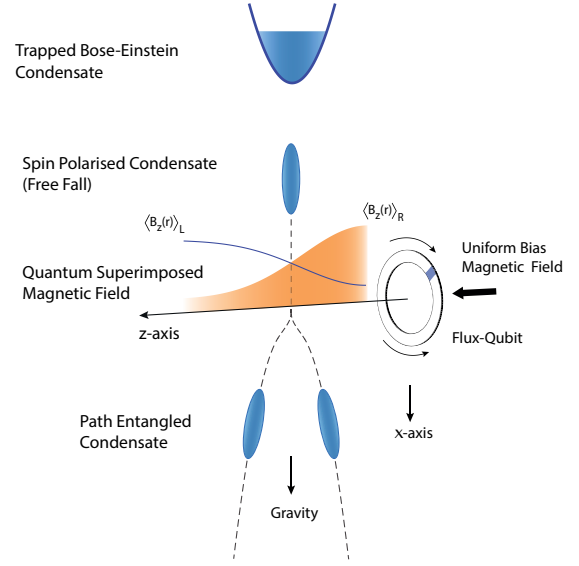


Fig. 1: A schematic of the quantum Stern-Gerlach experiment, where a freely falling spin polarised Bose-Einstein condensate is interacting with the quantum superimposed magnetic field of the flux-qubit. A uniform magnetic field is applied along the quantization axis and it also bias the flux-qubit such that the potential energy of the flux-qubit corresponds to a symmetric double well. The expectation value of the predominant component of the magnetic field,  $\langle B_z(r) \rangle$ , increases with the distance ( $z$ ) for the clockwise flow of the persistent current and decreases with the distance ( $z$ ) for the anti-clockwise flow of the persistent current. After interaction with the magnetic field of the flux-qubit, the Bose-Einstein condensate can be path entangled.

a quantum superposition of the persistent current flowing in the opposite directions through the flux-qubit loop produces a macroscopic quantum superposition of the magnetic flux. For further reference, an additional dimension of the flux-qubit has been explored in the reference [27].

The ground state wave-function of the flux-qubit is delocalised over the double-well potential therefore, the persistent current and the corresponding net magnetic flux passing through the loop of the flux-qubit are quantum superimposed. It is the quantum superposition of the magnetic flux which eventually produces a quantum superimposed magnetic field gradient. Consider an atom is far from the flux-qubit such that the interaction between the atom and the flux-qubit is zero. The flux-qubit is prepared in its ground quantum state  $\int C_g(\Phi, t)|\Phi\rangle d\Phi$ , where the quantum state is written in the net magnetic flux basis and  $C_g(\Phi, t)$  is the corresponding ground state wave-function. The quantum state of a free falling atom in the centre of mass frame is  $|\psi(\mathbf{r}; t)\rangle \sum_{m_F=-F}^F c_{m_F}(t)|m_F\rangle$ . Therefore, prior to the interaction the total quantum state of the atom and the field is a product state *i.e.*  $|\psi(\mathbf{r}; t)\rangle \sum_{m_F=-F}^F c_{m_F}(t)|m_F\rangle \int C_g(\Phi, t)|\Phi\rangle d\Phi$ . During the free fall, the atom comes in the close proximity of the flux-qubit and it interacts, for a time  $\Delta t$ , with the quan-

tum superimposed magnetic field of the flux-qubit. The interaction vanishes as the free falling atom moves away from the flux-qubit. For each value of the net magnetic-flux, there is a corresponding magnetic field gradient and a corresponding imparted transverse momentum. It is assumed that the net magnetic flux and the magnetic field gradient can be determined simultaneously. Therefore, after the interaction, the total quantum state of the atom and the field becomes

$$|\alpha(\mathbf{r}, \Phi, \mathbf{m}_F; t)\rangle_T = \sum_{m_F=-F}^F \left( \int \int \Psi(r, \Phi, m_F, t) C_g(\Phi, t) |r\rangle |\Phi\rangle dr d\Phi \right) \times c_{m_F}(t) |m_F\rangle \quad (2)$$

In general, the amplitude  $\Psi(r, \Phi, m_F, t)$  is an inseparable function of  $r$ ,  $\Phi$  and  $m_F$  therefore, the quantum state given in Eq. 2 is an entangled quantum state. For the quantum state given in Eq. 2 to be treated as an entangled quantum state, the momentum uncertainty of the initial spatial wave-function  $\psi(r, t)$  of the atom should be less than the uncertainty of the transverse linear momentum imparted to the atom *i.e.* the uncertainty of the momentum of wave-function  $\Psi(r, \Phi, m_F, t)$  along the transverse direction which is parallel to the quantization axis. Such an entangled quantum state is a macroscopic hybrid quantum state *i.e.* a Schrödinger-cat state, where the external degrees of freedom of the atom and the magnetic-flux are continuous variables, while the spin degree of freedom of the atom is discrete. For a given spin projection state  $|m_F\rangle$ , which has a non-zero interaction with the magnetic field, the term in the bracket of Eq. 2 is an entangled quantum state of the atomic path with the magnetic flux.

**Bose-Einstein condensate and quantum Stern-Gerlach experiment.** – Consider a schematic of a quantum Stern-Gerlach experiment as shown in Fig. 1 where a Bose-Einstein condensate of  $N$ -atoms is undergoing a free fall and it interacts with the magnetic field of the flux-qubit, which is prepared in its ground state. The Bose-Einstein condensate is spin polarised, where spins of all the atoms are aligned along the quantization axis *i.e.* all the atoms of the Bose-Einstein condensate are prepared in a given spin projection state  $|m_F\rangle$ , which has a non-zero coupling with the magnetic field. The interaction among the atoms of a freely falling Bose-Einstein condensate is assumed to be zero. The quantum state of  $N$ -atom Bose-Einstein condensate prior to its interaction with the flux-qubit can be written as  $(\int \psi(r, t) |r\rangle dr)^{\otimes N} \equiv \int \psi(r_1, t) |r_1\rangle dr_1 \otimes \int \psi(r_2, t) |r_2\rangle dr_2 \dots \otimes \int \psi(r_n, t) |r_n\rangle dr_n$  where the variables  $r_1, r_2$  upto  $r_n$  are the spatial coordinates of  $N$  atoms with same wavefunction  $\psi(r, t)$ . Since all the atoms of the Bose-Einstein condensate are influenced by the superimposed magnetic field therefore, after interaction with the flux-qubit, the total quantum state of the Bose-Einstein condensate becomes entangled with

the quantum state of the flux-qubit. Therefore, the total quantum state can be written as

$$|\beta(\mathbf{r}, \Phi; t)\rangle \propto \int \left( \int \Psi(r, \Phi, t) |r\rangle dr \right)^{\otimes N} C_g(\Phi, t) |\Phi\rangle d\Phi \quad (3)$$

Where,  $(\int \Psi(r, \Phi, t) |r\rangle dr)^{\otimes N} \equiv \int \Psi(r_1, \Phi, t) |r_1\rangle dr_1 \otimes \int \Psi(r_2, \Phi, t) |r_2\rangle dr_2 \dots \otimes \int \Psi(r_n, \Phi, t) |r_n\rangle dr_n$  is the quantum state of  $N$  non-interacting atom Bose-Einstein condensate entangled with the net magnetic flux passing through the loop of the flux-qubit and  $C_g(\Phi, t)$  is the ground state wave-function of the flux-qubit. The spin degree of freedom is omitted from Eq. 3 since the Bose-Einstein condensate is spin polarised. For a fixed value of the net magnetic flux passing through the flux-qubit loop, there is a corresponding imparted transverse linear momentum, which is same for all atom of the Bose-Einstein condensate. It is the imparted transverse linear momentum which is well defined for a given value of the net magnetic flux passing through the flux-qubit loop. However, there is a finite uncertainty in the transverse linear momentum due to a finite extension of the wave-function of the Bose-Einstein condensate. It is important to note that in the case of a semi-classical Stern-Gerlach experiment the spin polarised atoms, in a given quantum state  $|m_F\rangle$ , travel along a unique path (with a non-zero uncertainty). While, in the case of quantum Stern-Gerlach experiment, even the spin polarised atoms can travel along different distinct paths due to the quantum nature of the magnetic field.

The ground state of the symmetric double-well potential of the flux-qubit can be written as  $(|\Phi\rangle_L + |\Phi\rangle_R)/\sqrt{2}$ , where  $|\Phi\rangle_L$  and  $|\Phi\rangle_R$  are the quantum states corresponding to the persistent current flowing clockwise and anti-clockwise direction, respectively. Each potential well of the double-well potential is harmonic around their respective minima therefore, the corresponding ground state wave-function  $C_g(\Phi, t)$  of the double-well potential can be written as a sum of two Gaussian functions centred at magnetic flux values  $\Phi_L$  and  $\Phi_R$ , such that  $C_g(\Phi) \simeq (e^{-\frac{(\Phi-\Phi_L)^2}{2\Delta\Phi^2}} + e^{-\frac{(\Phi-\Phi_R)^2}{2\Delta\Phi^2}})/\sqrt{2\pi^{1/4}\Delta\Phi^{1/2}}$ , where  $\Delta\Phi$  is the width of each Gaussian. If the width  $\Delta\Phi$  is much less than the distance between the peaks of the wave-function *i.e.*  $\Delta\Phi \ll \Phi_R - \Phi_L$ , then  $|\Phi\rangle_R$  and  $|\Phi\rangle_L$  can almost be considered orthogonal to each other *i.e.*  ${}_L\langle\Phi|\Phi\rangle_R \simeq 0$ . For a flux-qubit, consisting of a circular super-conducting closed loop, the predominant component of magnetic field and the magnetic field gradient is along the  $z$ -axis. Consider the expectation value of the predominant component of the magnetic field gradient corresponding to the magnetic flux ground state  $|\Phi\rangle_L$  of the left potential well and the magnetic flux state  $|\Phi\rangle_R$  of the right potential well are  $\partial\langle B_z(r)\rangle_L/\partial z$  and  $\partial\langle B_z(r)\rangle_R/\partial z$ , respectively. The field and its gradient are time dependent with respect to the frame of reference of a freely falling Bose-Einstein condensate. Therefore, after being interacted with the flux-



qubit, for a time duration  $\Delta t$ , the expectation value of the imparted transverse linear momenta (along the  $z$ -axis) of each atom of the Bose-Einstein condensate corresponding to  $|\Phi\rangle_L$  and  $|\Phi\rangle_R$  are  $\langle p_z \rangle_L = -m_F g_F \mu_B \int_0^{\Delta t} \frac{\partial \langle B_z(r,t) \rangle_L}{\partial z} dt$  and  $\langle p_z \rangle_R = -m_F g_F \mu_B \int_0^{\Delta t} \frac{\partial \langle B_z(r,t) \rangle_R}{\partial z} dt$ , respectively. The imparted transverse linear momenta corresponding to the magnetic flux state of each potential well has a non-zero uncertainty ( $\Delta P_z$ ) around their corresponding expectation values due to a non-zero width  $\Delta \Phi$  of the flux-qubit ground state wave-function. This additional uncertainty of the transverse momenta can be ignored, if  $\Delta \Phi \ll \Phi_R - \Phi_L$  such that  $\Delta P_z \ll \Delta p_z$ , where  $\Delta p_z$  is the uncertainty of the  $z$ -component of the momentum of the Bose-Einstein condensate wave-function prior to its interaction with the flux-qubit.

The magnetic field from the flux-qubit also interacts with the environment, which consists of a substrate on which the flux-qubit is fabricated, and the measurement devices. Such a coupling with the environment produces decoherence of the flux-qubit quantum state due to its entanglement with the environment. Suppose a flux-qubit is prepared in its ground state and immediately the Bose-Einstein condensate enters the region of interaction. To produce a quantum entanglement of the Bose-Einstein condensate with the flux-qubit, the interaction of the Bose-Einstein condensate with the flux-qubit should complete prior to the decoherence of the flux-qubit quantum state *i.e.* the interaction time  $\Delta t$  must be considerably less than the decoherence time  $t_d$ . The time of interaction  $\Delta t$  is determined by the  $x$ -component of the velocity of the Bose-Einstein condensate. Therefore, three regimes of coupling can be classified. Where, (1.) The regime of strong coupling, if  $|\langle p_z \rangle_R - \langle p_z \rangle_L| \gg \Delta p_z$  where,  $|\langle p_z \rangle_R - \langle p_z \rangle_L| = |m_F g_F \mu_B (\int_0^{\Delta t} \frac{\partial \langle B_z(r,t) \rangle_L}{\partial z} dt - \int_0^{\Delta t} \frac{\partial \langle B_z(r,t) \rangle_R}{\partial z} dt)|$ . In this case, the Bose-Einstein condensate is quantum entangled with the quantum state of the flux-qubit *i.e.* the total quantum state given in Eq. 3 is considered to be an entangled quantum state. The interaction time  $\Delta t$  cannot be increased arbitrarily, in order to increase  $|\langle p_z \rangle_R - \langle p_z \rangle_L|$ , as the interaction time should be less than the decoherence time  $t_d$ . (2.) The regime of weak coupling, if  $|\langle p_z \rangle_R - \langle p_z \rangle_L| \sim \Delta p_z$ . In the case of weak coupling, the wave-functions of the Bose-Einstein condensate with expectation value of the imparted transverse momentum  $\langle p_z \rangle_R$  and  $\langle p_z \rangle_L$  are non orthogonal. In this case, the split paths of the Bose-Einstein condensate are partially overlapping. (3.) If  $|\langle p_z \rangle_R - \langle p_z \rangle_L| \ll \Delta p_z$ , the total quantum state given in Eq. 3 remains a product state.

The ground state of a symmetrically biased flux-qubit is  $|\Phi\rangle_g = (|\Phi\rangle_L + |\Phi\rangle_R)/\sqrt{2}$ , which implies that the corresponding ground state wave-function is  $C_g(\Phi) = (C_L(\Phi) + C_R(\Phi))/\sqrt{2}$ , where  ${}_L\langle \Phi | \Phi \rangle_R \simeq 0$ ,  $C_L(\Phi) = {}_L\langle \Phi | \Phi \rangle_g$  and  $C_R(\Phi) = {}_R\langle \Phi | \Phi \rangle_g$ . If the additional uncertainty  $\Delta P_z \ll \Delta p_z$  for  $\Delta \Phi \ll \Phi_R - \Phi_L$ , then the quantum state

given in Eq. 3 can be approximately written as

$$|\beta(\mathbf{r}, \Phi; t)\rangle \approx \frac{1}{\sqrt{2}} (|\Psi(\mathbf{r}; t)\rangle_1 |\Phi\rangle_L + |\Psi(\mathbf{r}; t)\rangle_2 |\Phi\rangle_R) \quad (4)$$

Where,  $|\Psi(\mathbf{r}; t)\rangle_1$  and  $|\Psi(\mathbf{r}; t)\rangle_2$  are the momentum imparted quantum states of the non-interacting  $N$ -atom Bose-Einstein condensate such that

$$\begin{aligned} |\Psi(\mathbf{r}; t)\rangle_1 &= \left( \int \Psi(r, \Phi_L, t) |r\rangle dr \right)^{\otimes N} \\ |\Psi(\mathbf{r}; t)\rangle_2 &= \left( \int \Psi(r, \Phi_R, t) |r\rangle dr \right)^{\otimes N} \end{aligned} \quad (5)$$

and

$$\begin{aligned} |\Phi\rangle_L &= \int C_L(\Phi, t) |\Phi\rangle d\Phi \\ |\Phi\rangle_R &= \int C_R(\Phi, t) |\Phi\rangle d\Phi \end{aligned} \quad (6)$$

#### Path entanglement of Bose-Einstein condensate.

– The interaction between the Bose-Einstein condensate and the flux-qubit diminishes as the freely falling Bose-Einstein condensate moves away from the flux-qubit magnetic field region. Consider, immediately after the interaction time  $\Delta t$ , a Hadamard operation is applied on the flux-qubit quantum state such that  $|\Phi\rangle_L \mapsto \frac{|\Phi\rangle_L + |\Phi\rangle_R}{\sqrt{2}}$  and  $|\Phi\rangle_R \mapsto \frac{|\Phi\rangle_L - |\Phi\rangle_R}{\sqrt{2}}$ . If the time duration to apply a Hadamard operation is  $t_h$  then  $\Delta t + t_h$  should be much less than the decoherence time  $t_d$ . Therefore, the quantum state given in Eq. 4, after applying the Hadamard operation on the flux-qubit quantum state, becomes

$$\begin{aligned} |\beta(\mathbf{r}, \Phi; t)\rangle &\approx \frac{1}{\sqrt{2}} \left( \frac{|\Psi(\mathbf{r}; t)\rangle_1 + |\Psi(\mathbf{r}; t)\rangle_2}{\sqrt{2}} \right) |\Phi\rangle_L \\ &+ \frac{1}{\sqrt{2}} \left( \frac{|\Psi(\mathbf{r}; t)\rangle_1 - |\Psi(\mathbf{r}; t)\rangle_2}{\sqrt{2}} \right) |\Phi\rangle_R \end{aligned} \quad (7)$$

After the Hadamard operation, the flux-qubit quantum state is measured in the magnetic flux basis. If the time required to perform this measurement is  $t_m$  then  $\Delta t + t_h + t_m \ll t_d$  *i.e.* the quantum state of the flux-qubit must be measured much earlier than its decoherence time. Therefore, if the measurement outcome is  $|\Phi\rangle_L$ , then the quantum state of atoms collapses to a path entangled quantum state of a Bose-Einstein condensate *i.e.*  $\frac{|\Psi(\mathbf{r}; t)\rangle_1 + |\Psi(\mathbf{r}; t)\rangle_2}{\sqrt{2}}$ . In a path entangled quantum state, the Bose-Einstein condensate of  $N$ -atoms behaves as a single particle whose paths are quantum superimposed. On the other hand, if the measurement outcome is  $|\Phi\rangle_R$ , then the quantum state of atoms collapses to a path entangled Bose-Einstein condensate of quantum state  $\frac{|\Psi(\mathbf{r}; t)\rangle_1 - |\Psi(\mathbf{r}; t)\rangle_2}{\sqrt{2}}$ . If the flux-qubit quantum state is not measured and correlated with the quantum state of the Bose-Einstein condensate or if the quantum state of the

flux-qubit is ignored then the quantum state of the Bose-Einstein condensate shall be a mixed state.

After completion of a measurement on the flux-qubit, the Bose-Einstein condensate is disentangled with the flux-qubit and the path entangled Bose-Einstein condensate continues falling freely under gravity. Since the atoms are falling in the interaction free region therefore, the path entanglement of Bose-Einstein condensate persists even for a time much larger than the decoherence time of the flux-qubit. Path entanglement of Bose-Einstein condensate can be detected by recombining the paths of the entangled condensate and by detecting all the atoms at a given location. As the position of the number detector is displaced an interference pattern can be obtained. Each time exactly the same number of atoms must be prepared in the path entangled state and all of the atoms must be detected [28]. However, the resulting interference pattern shall be contracted as compared to the interference pattern of two overlapping Bose-Einstein condensates as if the de Broglie wavelength of path entangled atoms is reduced such that  $\lambda_{path} = \lambda_{BEC}/N$ . Where,  $\lambda_{BEC}$  is the instantaneous wavelength of an atom from a falling Bose-Einstein condensate. In the case of path entangled Bose-Einstein condensate, all of the  $N$  noninteracting atoms behave like a single particle whose mass is  $N$  times the mass of an individual atom.

**Estimations.** – The decoherence time of the flux-qubit is a fundamental factor, which decides the strong coupling regime. To estimate the order of decoherence time required to produce a strong coupling, consider a flux-qubit of circular loop cross-section where, the inner radius of the loop is  $2.0 \mu\text{m}$ , the outer radius of the loop is  $2.5 \mu\text{m}$  and the thickness of the loop is  $1.0 \mu\text{m}$ . If the loop is fabricated on a non-magnetic material then the self inductance of the loop with given dimensions is  $L \simeq 6.44 \text{ pH}$ . A uniform magnetic field,  $650 \times 10^{-4} \text{ mT}$ , is applied along the  $z$ -axis to magnetically bias the flux-qubit at half of the flux quantum, where the flux quantum  $\Phi_0 = 2.0678 \times 10^{-15} \text{ Tm}^2$ . Consider minima of the symmetric double-well potential corresponding to the left and the right potential wells are located at  $0.25\Phi_0$  and  $0.75\Phi_0$ , respectively. For these minima values, the difference between the peaks ( $\Phi_R - \Phi_L$ ) of the ground state wave-function is  $\Phi_0/2$ . Assuming,  $\Delta\Phi \ll \Phi_R - \Phi_L$ , therefore, for the quantum state  $|\Phi\rangle_L$ , corresponding to the left potential well, the expectation value of the persistent current is  $-80.3 \mu\text{A}$ , which produces a magnetic field in the opposite direction to the applied bias magnetic field. Similarly, for the quantum state  $|\Phi\rangle_R$ , corresponding to the right potential well, the expectation value of the persistent current is  $+80.3 \mu\text{A}$  and it produces a magnetic field in the direction of the applied bias magnetic field. The Josephson junction critical current should be higher than the calculated expectation value of the persistent current. Corresponding to the ground state of the left potential well, the maximum of the expectation value of the magnetic field

gradient is  $\frac{\partial(B_z(r,t))_L}{\partial z} \sim 81.8 \times 10^{-1} \text{ Tm}^{-1}$  at a distance  $\sim 1.25 \mu\text{m}$  from the centre of the loop on the  $z$ -axis. Corresponding to the ground state of the right potential well, the direction of persistent current is reversed therefore, the minimum of the expectation value of the magnetic field gradient is  $\frac{\partial(B_z(r,t))_R}{\partial z} \sim -81.8 \times 10^{-1} \text{ Tm}^{-1}$  at a same distance  $\sim 1.25 \mu\text{m}$  from the centre of the loop on the  $z$ -axis. Around this point, the magnetic field gradient remains almost constant up to a radius of about  $2.0 \mu\text{m}$  in a plane parallel to the plane of the loop and up to a distance of about  $1.0 \mu\text{m}$  along the  $z$ -axis. Therefore, consider an anisotropic Bose-Einstein condensate just before it enters the magnetic field region of the flux-qubit. The Gaussian wave-function of the falling Bose-Einstein condensate has widths along the directions parallel to the  $z$ -axis and the  $y$ -axis to be  $1.0 \mu\text{m}$ . The free falling Bose-Einstein condensate can be guided through an atom waveguide up-to the interaction region in order to maintain its required extension in the  $y - z$  plane prior to its interaction with the flux-qubit. The waveguide potential can be turned off immediately when the Bose-Einstein condensate enters into the interaction region. The direction of the velocity of the Bose-Einstein condensate during the free fall is parallel to the  $x$ -axis. The width of the Bose-Einstein condensate along a direction parallel to the  $x$ -axis is considered to be  $5.0 \mu\text{m}$ . The position of the flux-qubit is adjusted such that during the free fall the Bose-Einstein condensate passes through the region of maximum magnetic field gradient, which is located at a distance  $\sim 1.25 \mu\text{m}$  from the centre of the loop. The momentum uncertainty of the Bose-Einstein condensate, prior to its interaction with the flux-qubit, along the quantization axis ( $z$ -axis) is  $\Delta p_z = \hbar/\Delta z$ , where  $\Delta z$  is the width of the wave-function of the Bose-Einstein condensate along a direction parallel to the  $z$ -axis. Therefore, in order to achieve the strong coupling regime *i.e.*  $|\langle p_z \rangle_R - \langle p_z \rangle_L| \gg \Delta p_z$ , the calculated time of interaction  $\Delta t$  between the Bose-Einstein condensate and the flux-qubit should be much greater than  $\sim 2.1 \mu\text{sec}$ . Where, the time limit,  $\sim 2.1 \mu\text{sec}$ , is calculated for the weak coupling regime. The time of interaction is calculated for the Bose-Einstein condensate of rubidium atom ( $^{87}\text{Rb}$ ) for a quantum state  $F = 2$  and  $m_F = +2$ , where  $g_F = 0.5$ . The  $x$ -component of the velocity of the free falling Bose-Einstein condensate is chosen to be such that the condensate remains in the high magnetic field gradient region for a time  $\Delta t$ , which should be much less than the decoherence time.

Therefore, in order to produce a macroscopic quantum entanglement of the Bose-Einstein condensate with the quantum state of the flux-qubit, the decoherence time of the flux-qubit should be much higher than the calculated value of the time of interaction *i.e.*  $t_d \gg 2.1 \mu\text{sec}$ . The decoherence time of a flux-qubit of the order of microseconds has been observed [21, 29]. However, the flux-qubit with dimensions described in this paper is considered to have a low self-inductance, which results a high

expectation value of the persistent current  $\sim 80.3 \mu\text{A}$ . The Josephson-junction of the flux-qubit should be able to pass through it more than the calculated value of the persistent current without any dissipation. Furthermore, to realize a path entangled Bose-Einstein condensate, the decoherence time of the flux-qubit should be such that  $t_d \gg 2.1 \mu\text{sec} + t_h + t_m$ .

**Conclusion.** – In this paper, a quantum Stern-Gerlach thought experiment and its fundamental significance has been presented. In addition to the intrinsic angular momentum of an atom the magnetic field and the magnetic field gradient are also treated quantum mechanically by incorporating quantum superposition principle. As a consequence, the path of atoms can split into more than one distinct paths even for the case of spin polarised atoms along the quantization axis. In contrast to the semiclassical Stern-Gerlach experiment, the quantum Stern-Gerlach experiment can produce a macroscopic quantum entanglement of the path of the Bose-Einstein condensate with the quantum state of the magnetic field. In addition, the quantum Stern-Gerlach experiment can produce a path entanglement of Bose-Einstein condensate where, the Bose-Einstein condensate can occupy physically distinct locations. The path entanglement of the Bose-Einstein condensate can persist for a time much larger than the decoherence time of the flux-qubit. In this paper, different regimes of coupling between the flux-qubit and the Bose-Einstein condensate are discussed.

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